QoS Satisfaction Games for Spectrum Sharing
-Technical Report

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Abstract—Today’s wireless networks are facing tremendous growth and many applications have more demanding quality of service (QoS) requirements than ever before. However, there is only a finite amount of wireless resources (such as spectrum) that can be used to satisfy these demanding requirements. We present a general QoS satisfaction game framework for modeling the issue of distributed spectrum sharing to meet QoS requirements. Our study is motivated by the observation that finding globally optimal spectrum sharing solutions with QoS guarantees is NP hard. We show that the QoS satisfaction game has the finite improvement property, and the users can self-organize into a pure Nash equilibrium in polynomial time. By bounding the price of anarchy, we demonstrate that the worst case pure Nash equilibrium can be close to the global optimal solution when users’ QoS demands are not too diverse.

I. INTRODUCTION

The number of wireless devices such as smart-phones continues to increase rapidly, while the amount of spectrum available for these devices remains limited. Moreover, many new wireless applications such as the high definition video streaming and the online interactive gaming are emerging, making the quality of service (QoS) of wireless users higher and more diverse. Thus there is an urgent need to study the issue of how to efficiently allocate the limited spectrum to satisfy the QoS demands of as many users as possible.

There are two different approaches towards handling this issue. The first approach is a centralized one, where a network operator optimizes the spectrum allocation to meet the users’ QoS requirements. This approach puts most of the implementation complexity on the operator’s side, and wireless devices do not need to be very sophisticated. However, as networks grow larger and more heterogeneous, this approach may not be suitable for the following two reasons. First, the QoS demands of wireless users are highly heterogeneous, which implies that the operator needs to gather massive amounts of information from users in order to perform the centralized optimization. Second, finding the system-wide optimal QoS satisfaction solution itself is computationally challenging – in fact we show that it is NP hard. It is hence difficult for the operator to compute the optimal solution to meet users’ real-time QoS demands. The alternative approach is a decentralized one, where each wireless user makes the spectrum access decision locally to meet its own QoS requirement, while taking the network dynamics and other users’ actions into consideration. This is feasible since new technologies like cognitive radio [1]–[3] give users the ability to scan and switch channels easily. The decentralized approach enables more flexible spectrum sharing, scales well with the network size, and is particularly suitable when users belong to multiple network operators.

We investigate the decentralized approach using the framework of congestion game theory. Rosenthal proposed the original congestion game [4] to model how selfish players share heterogenous resources. A player’s utility from using a resource depends on the congestion level of that resource, which is the number of players sharing that resource. Congestion games can model spectrum sharing when the players represent wireless users and resources represent channels. However, in many wireless channels users are often highly heterogenous. The congestion games with player-specific utility functions considered in [4] are more appropriate for this general scenario. Authors in [6]–[8] have adopted such game models for studying spectrum sharing problems. In [9]–[13], we considered how graphical congestion games can be used to model spectrum sharing. Here the players are represented as vertices in a graph, and each player only interacts with his neighbors. The graph models the interference relationships between the wireless users.

A common assumption in previous congestion game based spectrum sharing literature is that a user’s utility strictly increases with its received data rate (and hence strictly decreases with the congestion level). This is true, for example, when users are running elastic applications such as file downloading. However, there are many other types of applications with more specific QoS requirements, such as VoIP and video streaming. These inelastic applications can not work properly when their QoS requirements (such as target data rates) are violated, but do not obtain additional benefits when given more resources than needed. This kind of traffic is becoming increasingly popular over the wireless networks (e.g., mobile video traffic exceeded 50% percent of all wireless traffic in 2011 according to the report by Cisco [14]). This motivates us to study the QoS satisfaction game in this paper. Rather than assuming that users wish to increase their data rates whenever possible, we assume that each user has a fixed QoS requirement. If the requirement is satisfied, then the user has no inclination to change his choice of resource. The concept of focusing upon satisfaction rather than data rate maximization was inspired by [15], [16].

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We propose a new framework of QoS satisfaction games, where satisfaction of a user’s QoS requirements depends on its congestion level (i.e., how many users competing with the user for the same spectrum band). The central theme of our results is that allowing users to selfishly and distributively share the spectrum has many advantages over the centralized optimization approach. First of all, we show that selfish spectrum sharing is feasible (with Theorem 1), by proving that selfish players can quickly organize themselves into pure Nash equilibria. Second, we show that an efficient centralized optimization approach is often unfeasible (with Theorem 2), by proving that maximizing social welfare is an NP hard problem. Third, we show that in many cases the game based solution is close to optimal, by bounding the price of anarchy of our systems (with Theorem 3). Together these three results suggest that a decentralized approach towards QoS satisfaction will often be highly effective. We also consider the case where the users are homogenous, and show that the social optima are precisely the same as the pure Nash equilibria in this case (with Theorem 4).

II. QoS SATISFACTION GAME

A. Game model

A QoS satisfaction game is defined by a triple \((N, C, (Q_n(x))_{n\in N, x\in C}, (D_n)_{n\in N})\) where:

- \(N \triangleq \{1, \ldots, N\}\) is the set of wireless users, also referred as the players.
- \(C \triangleq \{1, \ldots, C\}\) is the set of real channels. Each user can access at most one real channel at a time. Furthermore, we use 0 to represent the virtual channel. This will be useful when a user’s QoS requirement cannot be satisfied due to limited resource, then the user can choose to cease its transmission to save power consumption (i.e., choose the virtual channel). In summary, each user/player has a strategy set \(\tilde{C} \triangleq \{0, 1, \ldots, C\}\). The strategy profile of the game is given as \(x = (x_1, x_2, \ldots, x_N) \in \tilde{C}^N\), where each user \(n\) chooses a (virtual or real) channel \(x_n \in \tilde{C}\).
- \(Q_n(\cdot)\) is a non-increasing function that characterizes user \(n\)’s data rate in terms of the congestion level \(I^c(\cdot)\). The congestion level \(I^c(\cdot) = |\{n \in N : x_n = c\}|\) in strategy profile \(x\) is the number of users who choose the real channel \(c\). Like in many studies of congestion games for spectrum sharing (e.g., \([3, 9, 17]\)), we assume that a user’s data rate depends on the number of contending users. This is true when the network adopts a medium access mechanism such as TMDA or CSMA (with the same user window size). We also allow user specific data rate functions, i.e., different users may have different \(Q_n(\cdot)\) even on the same channel \(c\). This will allow users to have different transmission technologies, choose different coding/modulation schemes, and experience different channel conditions.
- \(D_n \geq 0\) is the data rate required by user \(n\) to support its applications. For example, listening to an MP3 online will require a small \(D_n\), whereas watching a high definition streaming video requires a large \(D_n\).

User \(n\)’s utility in strategy profile \(x\) is

\[
U_n(x) = \begin{cases} 
1, & \text{if } x_n \neq 0 \text{ and } Q^*_n(I^x_n(x)) \geq D_n, \\
0, & \text{if } x_n = 0, \\
-1, & \text{if } x_n \neq 0 \text{ and } Q^*_n(I^x_n(x)) < D_n.
\end{cases}
\]

A satisfied user is a user \(n\) with a utility equal to 1, i.e., its received data rate \(Q^*_n(I^x_n(x))\) is above or equal to its QoS requirement \(D_n\). A dormant user is a user \(n\) choosing the virtual channel \(x_n = 0\). Such a dormant user gains no benefit for successful channel usage and receives no penalty for expending power, and so it gets a utility of \(U_n(x) = 0\). A suffering user is a user \(n\) with a utility of \(-1\), i.e., its received data rate \(Q^*_n(I^x_n(x))\) is lower than its QoS requirement \(D_n\). Such a suffering user expends power without gaining benefit, and so it gets a utility of \(U_n(x) = -1\).

It is worth noting that we can easily generalize our model by allowing a user \(n\) to receive a utility of \(u_n\) if it is satisfied, \(v_n\) if it is dormant, and \(t_n\) if it is suffering, where \(u_n > v_n > t_n\). Making this generalization does not affect the better response dynamics or the set of pure Nash equilibria discussed later on, because the preference orderings of the strategies in the generalized game are the same as in our current model.

Our results about convergence (Theorem 1) and computational complexity (Theorem 2) also remain true for games with generalized utility functions. However, since the generalized games allow different users to receive different utilities when satisfied, our result about price of anarchy (Theorems 3) may not hold with the generalized games. In this paper, we will restrict our attention to the utility choices of 1, 0 and \(-1\) for simplicity. The case of generalized utility functions will be further explored in a future work.

B. Key game concepts

Definition 1 (Social Welfare): The social welfare of a strategy profile \(x\) is the sum of all users’ utilities, i.e., \(\sum_{n=1}^N U_n(x)\).

Definition 2 (Social Optimum): A strategy profile \(x\) is a social optimum when it maximizes social welfare.

Definition 3 (Better Response Update): The event where a user \(n\) changes its choice of channel from \(x_n\) to \(c\) is a better response update if and only if \(U_n(c, x_{-n}) > U_n(x_n, x_{-n})\) where \(x_{-n} = (x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N)\) is the strategy profile of all users except \(n\).

Definition 4 (Pure Nash Equilibrium): A strategy profile \(x\) is a pure Nash equilibrium if no users at \(x\) can perform a better response update, i.e., \(U_n(x_n, x_{-n}) \geq U_n(c, x_{-n})\) for any \(c \in \tilde{C}\) and \(n \in N\).

Definition 5 (Finite Improvement Property): A game has the finite improvement property if any asynchronous better response update process (i.e., no more than one user updates

1Technically speaking, our game is weakly isomorphic to this generalized version.
2As shown later, under such a utility setting, the number of satisfied users at both social optima and Nash equilibria equals to the sum of all users’ utilities at these equilibria.
Data Rate $Q^c_n$

Demand $D_n$

Threshold $T^c_n$

Congestion $I^c$

Fig. 1. Illustration of the interference threshold from transformation.

C. Interference threshold form transformation

To ease exposition, we will introduce an equivalent interference threshold form for the QoS satisfaction game. The key idea is to consider the critical congestion threshold below which a user’s QoS demands can be met. Since the data rate function $Q^c_n(I^c)$ is non-increasing with the congestion level $I^c$, there must exist a critical threshold value $T^c_n$, such that $Q^c_n(I^c) \geq D_n$ if and only if the congestion level $I^c \leq T^c_n$ (see Figure 1 for an illustration). Formally, given a pair of $(Q^c_n, D_n)$, we can construct the threshold $T^c_n$ so that

- if $Q^c_n(I^c) < D_n$ for each $I^c \in \{1, \ldots, N\}$ then $T^c_n = 0$,
- if $Q^c_n(I^c) > D_n$ for each $I^c \in \{1, \ldots, N\}$ then $T^c_n = N + 1$,
- otherwise $T^c_n$ is equal to the maximum integer $I^c \in \{1, \ldots, N\}$ such that $Q^c_n(I^c) \geq D_n$.

These conditions guarantee that

$$Q^c_n(I^c) \geq D_n \iff I^c \leq T^c_n,$$

for each real channel $c$. We can then express a QoS satisfaction game $g = (\mathcal{N}, \mathcal{C}, (Q^c_n)_{n \in \mathcal{N}, c \in \mathcal{C}}, (D_n)_{n \in \mathcal{N}})$ in the interference threshold form $g' = (\mathcal{N}, \mathcal{C}, (T^c_n)_{n \in \mathcal{N}, c \in \mathcal{C}})$. The utility of user $n$ can be computed accordingly as

$$U(x) = \begin{cases} 1, & \text{if } x_n \neq 0 \text{ and } I^c_n(x) \leq T^c_n, \\ 0, & \text{if } x_n = 0, \\ -1, & \text{if } x_n \neq 0 \text{ and } I^c_n(x) > T^c_n. \end{cases}$$

(3)

The interference threshold form transformation reduces the size of parameters by replacing $(Q^c_n, D_n)$ with $T^c_n$. Moreover, the result in (4) ensures that the original game $g$ is equivalent to the game $g'$, since the utility $U_n(x)$ received by user $n$ in $g$ is the same as that received by user $n$ in $g'$ for every strategy profile $x$ and user $n$. For the rest of the paper, we will analyze the QoS satisfaction game in the interference threshold form.

III. PROPERTIES OF QoS SATISFACTION GAME

Now we explore the properties of QoS satisfaction games, including the existence of pure Nash equilibrium and the finite improvement property.

A. Characterization of pure Nash equilibria

First of all, it is easy to see that at a pure Nash equilibrium each user must be either satisfied or dormant. To see this, consider a strategy profile $x$ where a user $n$ is suffering (i.e., its utility is $-1$). Then user $n$ can do a better response update by changing its channel to $0$ and becoming dormant. The presence of a player that can do a better response update implies that $x$ is not a pure Nash equilibrium, in this case. Similarly, there are also no suffering users at a social optimum, because having a suffering user becoming dormant increases their utility without decreasing any other user’s utility. Next we show in Theorem 1 that every QoS satisfaction game has the finite improvement property. This guarantees the existence of a pure Nash equilibrium.

**Theorem 1**: Every $N$-user QoS satisfaction game has the finite improvement property. Any asynchronous better response updating process is guaranteed to reach a pure Nash equilibrium, within no more than $2N + 3N^2$ updates.

**Proof**: We define a function $\Phi(x) = \sum_{n \in \mathcal{N}} F_n(x)$ which maps each strategy profile $x$ to an integer. Here we have

$$F_n(x) = \begin{cases} 2T^c_n - I^c_n(x), & \text{if } x_n \neq 0, \\ 0, & \text{if } x_n = 0. \end{cases}$$

(4)

for each player $n \in \mathcal{N}$.

Next we show that the value of the function $\Phi$ increases by at least one with each better response update. Consider the generic case where the system is in a strategy profile $x$ and then some player $n' \in \mathcal{N}$ does a better response update, and changes his channel from $x_{n'} = c' \in \mathcal{C}$ to $d' \in \mathcal{C}$. Let $y = (x_1, \ldots, x_{n'-1}, d', x_{n'+1}, \ldots, x_N)$ denote the new strategy profile which results from this better response update. Now we can show that $\Phi(y) \geq \Phi(x) + 1$ for each of the three possible types of better response updates that player $n'$ can make (depending upon whether $c'$ and $d'$ represent real or virtual channels).

The first case we consider is where the active player $n'$ changing from a real channel $c' \neq 0$ to the virtual channel $d' = 0$ (the user stops using a bad channel). The change in $\Phi$ caused by this update is

$$\Phi(y) - \Phi(x) = F_{n'}(y) - F_{n'}(x) + \sum_{n \in \mathcal{N}, n \neq n'} F_n(y) - F_n(x).$$

(5)

Since $y_{n'} = d' = 0$ is the virtual channel, we have

$$F_{n'}(y) = 0.$$ 

(6)

Also, since $x_{n'} = c' \neq 0$ is a real channel, we have

$$F_{n'}(x) = 2T^c_n - I^c_n(x).$$

(7)

On channel $c'$, there are $I'(x) - 1$ users sharing the channel with user $n'$ in the strategy profile. Each of these users has $F_n(y) = F_n(x) + 1$, as the congestion level on this channel decreases. For another player $n$ that does not use channel $c'$ at $x$, we have $F_n(y) = F_n(x)$. It follows that

$$\sum_{n \in \mathcal{N}, n \neq n'} F_n(y) - F_n(x) = I'(x) - 1.$$ 

(8)

Substituting Equations (6), (7) and (8) into Equation (5) gives

$$\Phi(y) - \Phi(x) = 2I'(x) - 2T^c_n - I' - 1.$$ 

(9)
Since it is a better response update for \( n' \) to stop using \( c' \) we have \( U_{n'}(x) = -1 \), and so \( I^d(x) \geq T^d + 1 \). Combining this with Equation 9 yield \( \Phi(y) \geq \Phi(x) + 1 \), as required.

One must also consider the cases where \( c' = 0 \) and \( d' \neq 0 \) (i.e., the active player starts using a new real channel), and the case where \( c' \neq 0 \) and \( d' = 0 \). For each of these two cases we can show that \( \Phi(y) \geq \Phi(x) + 1 \) and \( \Phi(x) = 2 \). We deal with these cases below.

The second case we consider is where the active player \( n' \) changes from a virtual channel \( c' = 0 \) to a real channel \( d' \neq 0 \) (i.e., where the active player starts using a new real channel). The change in \( \Phi \) caused by this update is
\[
\Phi(y) - \Phi(x) = F_{n'}(y) - F_{n'}(x) + \sum_{n \in N: n \neq n'} F_n(y) - F_n(x).
\]
\[ (10) \]

Since \( x_{n'} = c' = 0 \) is the virtual channel, we have
\[
F_{n'}(x) = 0.
\]
\[ (11) \]

Also, since \( y_{n'} = d' \neq 0 \) is a real channel, we have
\[
F_{n'}(y) = 2T^d_n - I^d_c(y) = 2T^d_n - I^d_c(x) - 1.
\]
\[ (12) \]

On channel \( d' \), there are \( I^d_c(x) \) users sharing the channel with user \( n' \) in the strategy profile \( x \). Each of these users \( n \) has \( F_n(y) = F_n(x) - 1 \), as the congestion level on this channel increases. For another player \( n \) that does not use channel \( d' \) at \( x \), we have \( F_n(y) = F_n(x) \). It follows that
\[
\sum_{n \in N: n \neq n'} F_n(y) - F_n(x) = -I^d_c(x).
\]
\[ (13) \]

Substituting Equations 11, 12 and 13 into Equation 10 gives
\[
\Phi(y) - \Phi(x) = 2T^d_n - 2I^d_c(x) - 1.
\]
\[ (14) \]

Since it is a better response update for \( n' \) to start using \( d' \) we have \( U_{n'}(y) = 1 \), and so \( I^d_c(x) + 1 = I^d_c(y) \leq T^d_n \). Combining this with Equation 14 yields \( \Phi(y) \geq \Phi(x) + 1 \), as required.

In the third case, where \( c' \neq 0 \) and \( d' \neq 0 \) (i.e., where the active player swaps their current real channel, for a better one), we have \( U_{n'}(x) = -1 \) and \( U_{n'}(y) = 1 \). It follows that we can decompose our better response update (where \( n' \) switches from real channel \( c' \) to real channel \( d' \)), into a better response update like the first case (where \( n' \) turns off their channel \( c' \)), followed by a better response update like in the second case (where \( n' \) starts using real channel \( d' \)). Performing these two operations has the net effect of converting \( x \) to \( y \), and according to our arguments each operation increases the value of \( \Phi \) by at least one. It follows that \( \Phi(y) \geq \Phi(x) + 2 \), as required.

So we have shown that for each possible type of better response update, the value of \( \Phi \) increases by at least one.

Finally we show that \( \Phi \) is bounded above and below and hence the asynchronus better response will stop within a finite number of steps. Note that for any strategy profile \( z \) and any player \( n \), we have \( -N \leq -I^z_{m}(z) \leq F_{n}(z) \leq 2T_{n}^{\psi} \leq 2N + 2 \). It follows that \( -N^2 \leq \Phi(z) \leq 2N^2 + 2N \). Combined with the fact that each better response will increase \( \Phi \) by at least one, this implies that it shall take no more than \( 2N + 3N^2 \) better responses to increase the \( \Phi \) value from the minimum to the maximum. When the initial strategy profile corresponds to a \( \Phi \) value larger than \( -N^2 \), it will take less than \( 2N + 3N^2 \) steps to stop. This implies that when we evolve the system under better response updates, we must reach a strategy profile \( w \) from which no further better response updates can be performed, within \( 2N + 3N^2 \) time slots. Such a strategy profile \( w \) must be a pure Nash equilibrium by definition.

Theorem 1 implies that the general QoS satisfaction games (with heterogeneous channels and users) can self organize into a stable state within polynomial time.

B. Finding a social optimum is NP hard

Although Theorem 1 implies that pure Nash equilibria are easy to achieve, it turns out that finding a social optimum can be extremely challenging (as our next result implies).

**Theorem 2:** The problem of finding a social optimum of a QoS satisfaction game is NP hard.

**Proof:** Mathematically, we can formulate the problem of finding a social optimum (i.e., the QoS satisfaction problem) as follows.

\[
\max \sum_{c=1}^{M} \sum_{n=1}^{N} h_{c,n}
\]
\[ (15) \]

subject to \( \sum_{c=1}^{M} h_{c,n} \leq T_n^c + (N + 1)(1 - h_{c,n}), \forall c, n \)
\[ \sum_{c=1}^{M} h_{c,n} \leq 1, \forall n \]
\[ h_{c,n} \in \{0, 1\}, \forall c, n, \]

where \( h_{c,n} = 1 \) means that channel \( c \) contains user \( n \).

Before the discussion of computational complexity of QoS satisfaction problem, we first introduce a closely-related definition called 3-dimensional matching.

**Definition 6:** Let \( X, Y, \) and \( Z \) be finite, disjoint sets, and let \( T \) be a subset of \( X \times Y \times Z \). That is, \( T \) consists of triples \( (x, y, z) \) such that \( x \in X, y \in Y \), and \( z \in Z \). Now \( M \subseteq T \) is a 3-dimensional matching if the following holds: for any two distinct triples \( (x_1, y_1, z_1) \in M \) and \( (x_2, y_2, z_2) \in M \), we have \( x_1 \neq x_2, y_1 \neq y_2, \) and \( z_1 \neq z_2 \).

In the following, we also call \( (x, y, z) \in T \) as an edge. Then we introduce the 3-dimensional matching decision problem as: suppose that \( |X| = |Y| = |Z| = I \), given an input \( T \) with \( |T| \geq I \), decide whether there exists a 3-dimensional matching \( M \subseteq T \) with \( |M| = I \). The 3-dimensional matching decision problem is a well-known NP-complete problem [19]. Based on this, we now have the following complexity results.

We prove that QoS satisfaction problem is NP-hard, by showing that given an oracle for solving QoS satisfaction problem, the 3-dimensional matching decision problem can be solved in polynomial time.

From an instance of 3-dimensional matching \( (X, Y, Z, T) \) with \( |X| = |Y| = |Z| = I \) and \( |T| = J \geq I \), we can create an instance of QoS satisfaction problem as follows. The set of channels is \( T \) (i.e., each edge is a channel). Let \( \psi = X \cup Y \cup Z \). For each element \( n \in \psi \), we regard it as user \( n \). We also introduce a new user set \( \phi \) that consists of \( J - I \) additional users. Then we define the threshold value \( T_n^2 \) as follows.
any user \( n \) in set \( \psi \), we set \( T^m_n = 3 \) if \( n \) is a element of any edge in \( T \), i.e., \( \exists (x, y, n) \in T \) or \( \exists (x, n, z) \in T \) or \( \exists (n, y, z) \in T \), and \( T^m_n = 1 \) otherwise. For any user \( n \) in set \( \phi \), we set \( T^m_n = 1 \). Clearly, 3 users from \( \psi \) can stay in a channel if and only if they forms an edge in \( T \). Thus, channels with 3 users correspond to a matching in \( T \). Therefore, the QoS satisfaction problem has an optimal solution that the number of satisfied users is \( 3I + J - I = 2I + J \) (i.e., there are \( I \) channels with each channel having 3 users and there are \( J - I \) remaining channels with each channel having 1 user), if and only if the 3-dimensional matching has a matching of size \( I \).

It follows that, if we have an oracle to find the optimal solution for QoS satisfaction problem, then we can decide in a polynomial time \( O(1) \) whether there exists a 3-dimensional matching \( M \subseteq T \) such that \( |M| = I \).

Theorem 2 provides the major motivation for our game theoretic study, because it asserts that the centralized network performance optimization is fundamentally difficult. It therefore makes sense to explore decentralized alternatives such as game based spectrum allocation.

\[ \text{C. Price of anarchy} \]

Although Theorem 2 implies that finding a social optimal strategy profile can be fundamentally difficult, we do know from Theorem 1 that pure Nash equilibria can be found with relative ease. This naturally raises the question of how the social welfare of pure Nash equilibria compare with that of optimal social welfare. This implies that pure Nash equilibria can be fundamentally difficult, we do know from Theorem 1 that pure Nash equilibria can be found with relative ease. This naturally raises the question of how the social welfare of pure Nash equilibria compare with that of optimal social welfare. In other words, how much social welfare will be lost by allowing the users to organize themselves, rather than directing them to a social optimum?

To gain insight into this issue, we study the price of anarchy (PoA) \([20]\). Recall that \( \tilde{C}^N \) is the set of strategy profiles of our game. Let \( \Xi \subseteq \tilde{C}^N \) denote the set of pure Nash equilibria of our game. Note that Theorem 1 implies that \( \Xi \) is non-empty.

Now the \textbf{price of anarchy} is defined as

\[ \text{PoA} = \frac{\max\{\sum_{n=1}^{N} U_n(x) : x \in \tilde{C}^N\}}{\min\{\sum_{n=1}^{N} U_n(x) : x \in \Xi\}}, \]

is defined to be the maximum social welfare of a strategy profile, divided by the minimum social welfare of a pure Nash equilibrium. The social welfare of a system at a pure Nash equilibrium can be increased by at most PoA times by switching to a centralized solution.

Theorem 3: Suppose we have a QoS satisfaction game \((N, C, (T^N_n)_{n \in N, c \in C})\) where \( T^N_n \geq 1 \) for each user \( n \) and each real channel \( c \). The price of anarchy of this game satisfies

\[ \text{PoA} \leq \min \left\{ N, \frac{\max\{T^N_n : n \in N, c \in C\}}{\min\{T^N_n : n \in N, c \in C\}} \right\}. \]

\[ \text{Proof:} \]

\[ 3 \text{In our case the PoA equals the maximum number of users that can be satisfied, divided by the minimum number of users that can be satisfied at a pure Nash equilibrium.} \]

\[ 4 \text{This constraint insures that some user will be satisfied in every pure Nash equilibrium of the game, and avoids the possibility of the PoA involving ‘division by zero’.} \]

Let \( B(x) \) denote the number of satisfied players in state \( x \).

Now if \( x \) is an social optimum or a pure Nash equilibrium then \( x \) has no suffering users, and so \( B(x) = \sum_{n=1}^{N} U_n(x) \).

This implies

\[ \text{PoA} = \frac{\max\{B(x) : x \in \tilde{C}^N\}}{\min\{B(x) : x \in \Xi\}} \]

Note that \( 0 \leq B(x) \leq N \), for any strategy profile \( x \). Now if \( \min\{B(x) : x \in \Xi\} = N \) then every pure Nash equilibrium is a social optimal, and so we have \( \max\{B(x) : x \in \tilde{C}^N\} = N \) and \( \text{PoA} = 1 \) (which clearly satisfies the stated conditions).

Now instead suppose \( \min\{B(x) : x \in \Xi\} < N \).

Let \( x^* \) be an optimum strategy profile. Now \( B(x^*) = \max\{B(x) : x \in \tilde{C}^N\} \leq N \). Also \( B(x^*) \) is equal to the number of users of channels in \( \{1, 2, ..., C\} \), under \( x^* \), and so there must be a channel \( c' \in \{1, 2, ..., C\} \) with at least \( \frac{B(x^*)(c)}{C} \) users (i.e., \( \exists c' \neq 0 : I^c(x^*) \geq \frac{B(x^*)}{C} \)). To see this, note that \( B(x^*) = \sum_{c=1}^{C} I^c(x^*) \leq C I^c(x^*) \), where \( c' \) is the real channel with the most users.

Now if \( I^c(x^*) = 0 \) then \( \frac{B(x^*)}{C} \leq I^c(x^*) = 0 \leq \max\{T^N_n : n \in N, c \in \{1, 2, ..., C\}\} \) clearly holds. Also, if \( I^c(x^*) \neq 0 \) then there will be some user \( n' \in N \) such that \( x^*_{n'} = c' \), and this user will not be suffering, and since they will not be suffering, we have \( \frac{B(x^*)}{C} \leq I^c(x^*) \leq T^c_{n'} \leq \max\{T^N_n : n \in N, c \in \{1, 2, ..., C\}\} \).

So now we have shown that \( B(x^*) = \max\{B(x) : x \in \tilde{C}^N\} \) adheres to the constraints \( B(x^*) \leq N \) and \( B(x^*) \leq C \max\{T^N_n : n \in N, c \in \{1, 2, ..., C\}\} \).

Let \( y^* \in \Xi \) be a pure Nash equilibrium such that \( B(y^*) = \min\{B(x) : x \in \Xi\} \). Now since \( T^N_n > 0, \forall n, \forall c \), we have that \( B(y^*) \geq 1 \). Also, since we are assuming \( B(y^*) = \min\{B(x) : x \in \Xi\} < N \), we have there is some dormant user \( n'' \in N \) such that \( y^*_{n''} = 0 \).

Now, since \( y^* \) is a pure Nash equilibrium, since for each real channel \( c' \in \{1, 2, ..., C\} \), that player \( n'' \) cannot benefit from switching to \( c' \). This implies \( I^c(y^*) \geq T^c_{n''} \geq \min\{T^N_n : n \in N, c \in \{1, 2, ..., C\}\} \).

So now we have shown that \( B(y^*) = \min\{B(x) : x \in \Xi\} \) is such that \( B(y^*) \geq 1 \) and \( B(y^*) \geq C \min\{T^N_n : n \in N, c \in \{1, 2, ..., C\}\} \).

Now since \( \max\{B(x) : x \in \tilde{C}^N\} \leq N \) and \( \min\{B(x) \in \Xi\} \leq 1 \) we have

\[ \text{PoA} \leq \frac{\max\{B(x) : x \in \tilde{C}^N\}}{\min\{B(x) : x \in \Xi\}} \leq N \]

Similarly, since \( \max\{B(x) : x \in \tilde{C}^N\} \leq C \max\{T^N_n : n \in N, c \in \{1, 2, ..., C\}\} \) and \( \min\{B(x) : x \in \Xi\} \leq C \min\{T^N_n : n \in N, c \in \{1, 2, ..., C\}\} \) we have

\[ \text{PoA} \leq \frac{C \max\{T^N_n : n \in N, c \in \{1, 2, ..., C\}\}}{C \min\{B(x) : x \in \tilde{C}^N, x \in \Xi\}} \]

One may cancel the Cs to get the stated result.
homogenous users, and two real channels with thresholds $T_1 = 1$ and $T_2 = 2$. The point $[I^0, I^1, I^2]$ represents the scenarios where there are $I^0$, $I^1$, and $I^2$ users on channels 0, 1, and 2 respectively. Every arrow represents a potential state transition that can occur due to a best response update. Note that we only keep track of the number of users of each channel, and that individual points may correspond to multiple strategy profiles. For example, the point $[I^0, I^1, I^2] = [1, 4, 0]$ corresponds to each strategy profile in the set $\{(0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1)\}$. The unique point corresponding to the pure Nash equilibria is shaded yellow.

Theorem 3 implies that the performance of every pure Nash equilibrium will be close to optimal when the minimum threshold of a user-channel pair is close to the maximum threshold of a user-channel pair. This is a quite nice result, when one considers that pure Nash equilibria can be quickly reached by better response updates (Theorem 1) while finding social optima is NP hard (Theorem 2).

D. QoS satisfaction games with homogenous users

Motivated by Theorem 3, we examine the special case of QoS satisfaction games with homogenous users. In this case social optima which are pure Nash equilibria can easily be found. We say that a QoS satisfaction game has homogenous users when $T^c_1 = T^c_2 = \ldots = T^c_N$, for each $c \in C$, i.e., each user has the same threshold for any real channel $c$. This corresponds to the case that all users have the same data rate function $Q^c_n$ on the same channel $c$ (but they may have different data rates on different channels) and the same QoS requirement $D_n$. For example, spectrum sharing in a network of RFID tags on a warehouse may correspond to such a QoS satisfaction game, because every device experiences the same environment and requires a similar data rate to operate.

When discussing QoS satisfaction games with homogenous users, we drop the subscripts and use $T^c$ to denote the common threshold for all users on channel $c$. Since users are homogenous, we only need to track of how many users choose each channel in order to describe the game dynamics (see Figure 2). Next we will show that any pure Nash equilibrium in QoS satisfaction games with homogenous users is also a social optimum.

**Theorem 4:** Let $x$ be a strategy profile of a QoS satisfaction game with $N$ homogenous users and $C$ real channels, with thresholds $T^1, T^2, \ldots, T^C$. The following three statements are equivalent:

1) $x$ is a pure Nash equilibrium;
2) $x$ is a social optimum;
3) There are no suffering users in $x$ and the number of satisfied users is $\min\{N, \sum_{c=1}^C T^c\}$.

**Proof:** Let $B(x) = \{n \in N : U_n(x) = 1\}$ denote the number of satisfied users in a strategy profile $x$. We will show that statement (1) implies statement (2), which in turn implies statement (3), which in turn implies statement (1).

Suppose statement (1) holds. Now we must have $U_n(x) \in \{0, 1\}$ for each $n \in N$. If $U_n(x) = 1$, for each $n \in N$ then $B(x) = N$. Now let us consider the case where we do not have $U_n(x) = 1$, for each $n \in N$. In this case there must exist some $n^* \in N$ such that $U_n^*(x) = 0$. Now $I^c(x) \geq T^c_n$ for each $c \in \{1, 2, \ldots, C\}$. In fact we must have $I^c(x) = T^c$ for each $c \in \{1, 2, \ldots, C\}$, because otherwise, if there existed a $c^* \in \{1, 2, \ldots, C\}$ such that $I^c_n(x) > T^c_c$, then there would have to be some user $n^* \in N$ such that $x_{n^*} = c^*$, $I^c_n(x) > T^c_{n^*}$ and $U_n^*(x) = -1$. This would contradict our requirement that $U_n(x) \in \{0, 1\}$ for each $n \in N$. The presence of this contradiction implies that, in fact, it must be that $I^c_n(x) = T^c$ for each $c \in \{1, 2, \ldots, C\}$. Now we have $B(x) = \{n \in N : x_n \neq 0\} = \sum_{c=1}^C I^c_n(x) = \sum_{c=1}^C T^c$. This shows that statement (2) holds.

Now instead suppose statement (2) holds. If $B(x) = N$ then $x$ is clearly a social optimal. Suppose instead that $B(x) = \sum_{c=1}^C T^c < N$. Now since $x$ has no suffering users we must have $B(x) = \{n \in N : x_n \neq 0\} = \sum_{c=1}^C I^c_n(x) = \sum_{c=1}^C T^c$. Now we shall prove $x$ is a social optimal by contradiction. If $x$ were not a social optimal, then there would have to be a social optimal $y$ with $B(y) = \sum_{c=1}^C I^c(y) = \sum_{c=1}^C T^c = B(x)$, but this implies that there exists some $c^* \in \{1, 2, \ldots, C\}$ such that $I^c_n(y) > T^c$ and so there must exist some $n^* \in N$ such that $x_{n^*} = c^*$ and $U_n^*(y) = -1$ and so $y$ is in fact not a social optimal. The presence of this contradiction shows that in fact no such $y$ can exist, and so $x$ is a social optimal, and we have shown statement (3).

Now instead suppose statement (3) holds. In this case we will show that $x$ is a pure Nash equilibrium by contradiction. Suppose $x$ is not a pure Nash equilibrium. Now there must exist a player $n^* \in N$ such that $U_n^*(x) = 0$ and a channel $c^* \neq 0$ such that $I^c_n(x) < T^c$. Now let $y$ be the strategy profile obtained by taking $x$ and having player $n^*$ do a better response update, by switching to channel $c^*$. Now $B(y) > B(x)$, and this contradicts our assumption (3) that $x$ is a social optimal. It follows that, in fact, (3) implies that $x$ is a pure Nash equilibrium. So (3) implies (1).

Theorem 3 implies that each pure Nash equilibrium is a social optimal in the special case where resources and users are homogenous. Theorem 4 is more than just a corollary of Theorem 3 since it states that this remains true when only the users are homogenous, and also states that the converse holds. Theorems 1 and 4 together imply that asynchronous better response updating will always converge to a social optimum in polynomial time when the game has homogenous users.
Moreover, Theorem 4 implies that when $\sum_{c=1}^{C} T_c \geq N$, there exists a satisfaction equilibrium (as defined in [15]) where all the users can be satisfied.

IV. CONCLUSION

In this paper, we proposed a framework of QoS satisfaction games to model the distributed QoS satisfaction problem among wireless users. The game based solution is motivated by the observation that computing the global optimal solution is an NP hard problem. We have explored several aspects of QoS satisfaction games including the convergence dynamics and the price of anarchy, and our results reveal that selfish spectrum sharing can be a very effective way to allow users to meet their QoS requirements. In the future, we will also look at the case where the channels are homogenous, and our preliminary results suggest that it is possible to design fast algorithms to find social optima. We shall also consider QoS satisfaction games on graphs - which take account of spatial reuse by using a graph to represent which users are close enough to interfere with one another. Also, we shall examine the scenario where users can access multiple channels simultaneously.

REFERENCES